DATE:

# 弹性力学.

## 二. 应变分析.

### 2.3 应变.

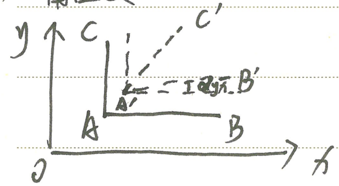
〈1〉 $A:(x, y)$ $B(x+dx, y)$

变形后 $A':(x+u, y+v)$ $B':(x+dx+u(x+dx, y), y+v(x+dx, y))$

$\delta(dx) = u(x+dx, y) - u(x, y) = \frac{\partial u}{\partial x}dx$.

$\therefore \varepsilon_{xx} = \frac{\partial u}{\partial x}$.

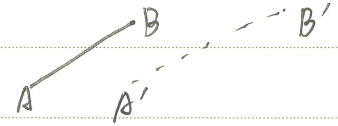
〈2〉 角应变




$\alpha_{yx} \approx \tan\alpha_{yx} = \dfrac{\frac{\partial v}{\partial x}dx}{dx+\frac{\partial u}{\partial x}dx} \approx \frac{\partial v}{\partial x}$.

$\gamma_{xy} = \gamma_{yx} = \alpha_{yx} + \alpha_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

〈3〉 任意两点的变化.




$A(x, y, z)$

$B(x+dx, y+dy, z+dz)$.

$A'(x+u, y+v, z+w)$

$B'(x+dx+u', y+dy+v', z+dz+w')$

$\begin{cases} u' = u(x+dx, y+dy, z+dz). \\ \cdots \\ \cdots \end{cases}$

泰勒展开：$u' = u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \cdots$

$v' = \cdots$

$w' = \cdots$

$$u' = u + \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz$$

令 $\omega_x = 2p = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$　　$\omega_x$：刚体转动分量

$\omega_y = 2q = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$　　$p,q,r$：刚体转动角度

$\omega_z = 2r = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

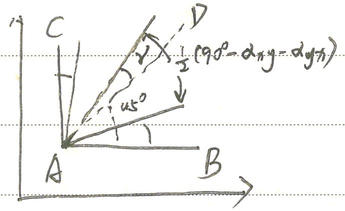
则有 $u' = u + e_x dx + \frac{1}{2}\gamma_{xy}dy + \frac{1}{2}\gamma_{zx}dz - r\,dy + q\,dz$

$v' = v + \frac{1}{2}\gamma_{xy}dx + e_y dy + \frac{1}{2}\gamma_{yz}dz - p\,dz + r\,dx$

$w' = w + \frac{1}{2}\gamma_{zx}dx + \frac{1}{2}\gamma_{yz}dy + e_z dz - q\,dx + p\,dy$

平移　变形　　转动

在二维情况下.




$\gamma = \frac{1}{2}(90^\circ - \alpha_{xy} - \alpha_{yx}) - 45^\circ + \alpha_{yx}$

$= \frac{1}{2}(\alpha_{yx} - \alpha_{xy})$

$= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

实际上，位移矢量的旋度便是刚体的转动.

$$\nabla \times (\vec{i}u + \vec{j}v + \vec{k}w) = \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \vec{i}\omega_x + \vec{j}\omega_y + \vec{k}\omega_z$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$





如果没有转动，则位移有一个应变函数 $d\Phi = u\,dx + v\,dy + w\,dz$

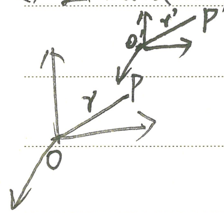
如果 $u, v, w$ 是 $x, y, z$ 的线性函数，则这种变形叫做均匀变形.

$u = u_0 + C_{11}x + C_{12}y + C_{13}z$

$v = v_0 + C_{21}x + C_{22}y + C_{23}z$

$w = w_0 + C_{31}x + C_{32}y + C_{33}z$

<> 定向线段的应变




$r'^2 = [\Delta x + u(p) - u(o)]^2 + [\Delta y + v(p) - v(o)]^2 + [\Delta z + w(p) - w(o)]^2$

$e = \frac{r' - r}{r}$　　$r' = (1+e)r$

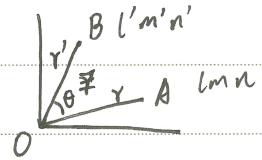
$(r + re)^2 = \cdots$

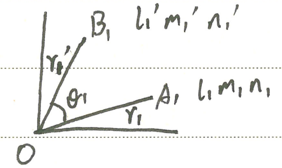
$(l, m, n)$ 为OP的方向余弦

$(1+e)^2 = \cdots$

$e = e_x l^2 + e_y m^2 + e_z n^2 + \gamma_{xy}lm + \gamma_{yz}mn + \gamma_{zx}nl$

<> 两定向线段的角度




$l_1 = \frac{(r_1)_x}{r_1} = \left(1 - e + \frac{\partial u}{\partial x}\right)l + \frac{\partial u}{\partial y}m + \frac{\partial u}{\partial z}n.$

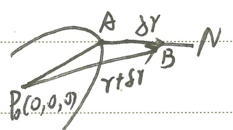
$\cos\theta = ll' + mm' + nn'$

$\cos\theta_1 = l_1l_1' + m_1m_1' + n_1n_1'$

## 2.4 应变二次曲面和主应变

令 $\pm K^2 = e r^2$，K是一个和长度有关的常数。

应变二次曲面 $e_x x^2 + e_y y^2 + e_z z^2 + \gamma_{xy} xy + \gamma_{yz} yz + \gamma_{xz} xz = \pm K^2$




曲面上任意一点的矢径 $P_0A$ 在变形后，A点是沿着曲线法线方向移动。

证：

令 $G = e_x x^2 + e_y y^2 + e_z z^2 + \gamma_{xy} xy + \gamma_{yz} yz + \gamma_{xz} xz$ 等于 $K^2$

则A点的法线方向与 $\nabla G$ 的方向一致。

$$N = \lambda \nabla G = \lambda\left(\vec i \frac{\partial G}{\partial x} + \vec j \frac{\partial G}{\partial y} + \vec k \frac{\partial G}{\partial z}\right)$$

其中 $\frac{\partial G}{\partial x} = 2e_x x + \gamma_{xy} y + \gamma_{zx} z$

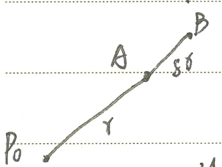
令 $\delta = \vec i \Delta x + \vec j \Delta y + \vec k \Delta z$ 则（考虑纯应变，无刚体转动）.

$$\Delta x = u(A) - u(P_0) = e_x x + \frac{1}{2}\gamma_{xy} y + \frac{1}{2}\gamma_{zx} z$$

则 $\Delta x : \Delta y : \Delta z = \frac{\partial G}{\partial x} : \frac{\partial G}{\partial y} : \frac{\partial G}{\partial z}$

### （二）主应变

假设变形后某一方向的线段 $P_0A$ 只沿他原来的方向伸长，




$P_0A\ (l, m, n)$　　$AB\ (l', m', n')$

$$\frac{l'}{l} = \frac{m'}{m} = \frac{n'}{n} = 1$$

$$l' = \frac{u(A) - u(P_0)}{\delta r} = \frac{e_x x + \frac{1}{2}\gamma_{xy} y + \frac{1}{2}\gamma_{zx} z}{e r}$$

又 $l = \frac{x}{r}$　$m = \frac{y}{r}$　$n = \frac{z}{r}$





$$\therefore \begin{cases} (e_x - e)x + \frac{1}{2}\gamma_{xy} y + \frac{1}{2}\gamma_{zx} z = 0 \\ \frac{1}{2}\gamma_{xy} x + (e_y - e) y + \frac{1}{2}\gamma_{yz} z = 0 \\ \frac{1}{2}\gamma_{zx} x + \frac{1}{2}\gamma_{yz} y + (e_z - e) z = 0 \end{cases}$$

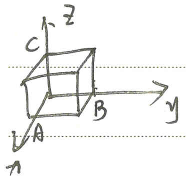
方程有非零解 $\therefore \begin{vmatrix} e_x - e & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & e_y - e & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & e_z - e \end{vmatrix} = 0.$

有 $-e^3 + \theta_1 e^2 - \theta_2 e + \theta_3 = 0$.

$$\text{其中} \begin{cases} \theta_1 = e_x + e_y + e_z \\ \theta_2 = e_x e_y + e_y e_z + e_z e_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \\ \theta_3 = e_x e_y e_z + \frac{1}{4}(\gamma_{xy}\gamma_{yz}\gamma_{zx} - e_x\gamma_{yz}^2 - e_y\gamma_{zx}^2 - e_z\gamma_{xy}^2) \end{cases}$$

三个根均为实根（P32）.（反证：假设两虚根）.

### （三）膨胀系数




$$\theta = \frac{dV}{V}$$

$$A' = \left(a + \frac{\partial u}{\partial x} a, \frac{\partial v}{\partial x} a, \frac{\partial w}{\partial x} a\right)$$

$$V + dV = \begin{vmatrix} 1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 1 + \frac{\partial w}{\partial z} \end{vmatrix} abc = V(1 + e_x + e_y + e_z)$$

$$\theta = e_x + e_y + e_z$$

不可压缩条件 $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.

在无旋（纯变形）下，应变函数 $\Phi$ 须满足 $\nabla^2 \Phi = 0$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

拉普拉斯方程。

2.5 协调方程 ← 描述应变分量之间的关系.

2.6 有限变形.

描述连续介质变形 { 拉格朗日：用变形前坐标作变量.
欧拉：用变形后坐标作变量





三.应力分析.

3.3 应力张量.

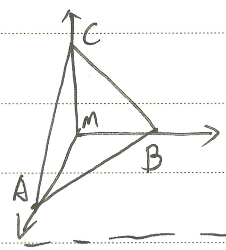
$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

3.4 平衡方程

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = \rho \frac{\partial^2 w}{\partial t^2} \end{cases}$$

6个未知数 $\sigma_x(x,y,z,t)$ $\tau_{xy}$
$\sigma_y$ $\tau_{yz}$
$\sigma_z$ $\tau_{xz}$

由力矩平衡有：
$$\begin{cases} \tau_{yz} = \tau_{zy} \\ \tau_{xy} = \tau_{yx} \\ \tau_{zx} = \tau_{xz} \end{cases}$$




ABC面上的力F$(X_N, Y_N, Z_N)$ =
$$\begin{cases} X_N = \sigma_x l + \tau_{yx} m + \tau_{zx} n \\ Y_N = \tau_{xy} l + \sigma_y m + \tau_{zy} n \\ Z_N = \tau_{xz} l + \tau_{yz} m + \sigma_z n \end{cases}$$

$$\iint_S f_x l + f_y m + f_z n \, dS = \iiint_\tau \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \, d\tau.$$ 高斯积分.

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四.应力和应变关系.

4.1 应力与应变~~的~~关系的最一般形式:

$$\begin{cases} \sigma_x = f_1(e_x, e_y, e_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}) \\ \sigma_y = f_2(e_x, e_y, e_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}) \\ \vdots \\ \tau_{zx} = f_6(e_x, e_y, \cdots \gamma_{zx}) \end{cases}$$

当应变很小时, $e, \gamma \ll 1$, 用~~泰~~Taylor展开:

$$\sigma_x = \frac{\partial f_1}{\partial e_x} e_x + \frac{\partial f_1}{\partial e_y} e_y + \cdots + \frac{\partial f_1}{\partial \gamma_{zx}} \gamma_{zx}.$$

$$\Rightarrow \begin{cases} \sigma_x = C_{11} e_x + C_{12} e_y + C_{13} e_z + C_{14} \gamma_{yz} + C_{15} \gamma_{zx} + C_{16} \gamma_{xy} \\ \sigma_y = C_{21} e_x + C_{22} e_y + C_{23} e_z + C_{24} \gamma_{yz} + C_{25} \gamma_{zx} + C_{26} \gamma_{xy} \\ \vdots \\ \tau_{xy} = C_{61} e_x + C_{62} e_y + C_{63} e_z + C_{64} \gamma_{yz} + C_{65} \gamma_{zx} + C_{66} \gamma_{xy}. \end{cases}$$

$C_{mn}$ 是弹性系数. 其中 $C_{ij} = C_{ji}$ 即 $C_{mn}$ 有 21个 独立量.

各向同性弹性体有2个 　　由应变能导证明.

4.2 功和应变能.

4.3 各向同性胡克定律.



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假如材料对于 $Oxy$ 平面对称, 13个独立的弹性系数.

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{pmatrix}$$

若关于 $Oxy$, $Oyz$ 对称, 9个独立的弹性系数.

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

沿着^{任意}两个互相垂直的方向看,弹性系数不变则有3个~~独~~独立的弹性系数.

$$\begin{pmatrix} C_1 & C_2 & C_2 & & & \\ C_2 & C_1 & C_2 & & & \\ C_2 & C_2 & C_1 & & & \\ & & & C_3 & & \\ & & & & C_3 & \\ & & & & & C_3 \end{pmatrix}$$

<2> 沿着成任意角度的两个方向看,弹性系数不变, 则有2个独立的弹性系数.

令 $\theta = e_x + e_y + e_z$ 则有

$$\begin{cases} \sigma_x = \lambda\theta + 2\mu e_x \quad \sigma_y = \lambda\theta + 2\mu e_y \quad \sigma_z = \lambda\theta + 2\mu e_z \\ \tau_{yz} = \mu \gamma_{yz} \quad \tau_{zx} = \mu \gamma_{zx} \quad \tau_{xy} = \mu \gamma_{xy} \end{cases}$$

$\lambda$ 和 $\mu$ 为拉梅常数.

剪切模量.

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令 $\Theta=\sigma_x+\sigma_y+\sigma_z=(3\lambda+2\mu)\theta$ 则以应力表示应变的胡克定律:

$$\begin{cases} e_x=\frac{\sigma_x}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\Theta \\ e_y=\frac{\sigma_y}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\Theta \\ e_z=\frac{\sigma_z}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\Theta \\ \gamma_{yz}=\frac{1}{\mu}\tau_{yz} \\ \gamma_{zx}=\frac{1}{\mu}\tau_{zx} \\ \gamma_{xy}=\frac{1}{\mu}\tau_{xy}. \end{cases}$$

## 4.4 弹性常数的测定

各向同性体的弹性常数:

(1) 简单拉伸实验.

$\sigma_x=T$ $\sigma_y=\sigma_z=\tau_{xy}=\tau_{yz}=\tau_{xz}=0.$

由上式有
$$\begin{cases} e_x=\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}T \\ e_y=e_z=\frac{-\lambda}{2\mu(3\lambda+2\mu)}T \quad ① \\ \gamma_{xy}=\gamma_{yz}=\gamma_{xz}=0. \end{cases}$$

由拉伸实验
$$\begin{cases} e_x=\frac{1}{E}T \quad e_y=e_z=-\frac{\nu}{E}T \quad ② \\ \gamma_{xy}=\gamma_{yz}=\gamma_{xz}=0. \end{cases}$$

E是杨氏模量, $\nu$是泊松比.




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由①和②有 $\begin{cases} E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \end{cases}$ $\nu=\frac{\lambda}{2(\lambda+\mu)}$

$\therefore$ $\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}$ $\mu=\frac{E}{2(1+\nu)}$

由 $\lambda>0$ $\mu>0$ $E>0$ 有 $0\le\nu\le\frac{1}{2}$.

$$\therefore \begin{cases} e_x=\frac{1}{E}(\sigma_x-\nu(\sigma_y+\sigma_z)) \\ e_y=\frac{1}{E}(\sigma_y-\nu(\sigma_z+\sigma_x)) \\ e_z=\frac{1}{E}(\sigma_z-\nu(\sigma_x+\sigma_y)) \\ \gamma_{yz}=\frac{2(1+\nu)}{E}\tau_{yz} \\ \gamma_{zx}=\frac{2(1+\nu)}{E}\tau_{zx} \\ \gamma_{xy}=\frac{2(1+\nu)}{E}\tau_{xy}. \end{cases}$$

(2) 纯剪试验 (即通过圆柱体纯扭转试验).

$\tau_{xy}=\tau=$常数 $\sigma_x=\sigma_y=\sigma_z=\tau_{yz}=\tau_{zx}=0.$

$\Longrightarrow \gamma_{xy}=\frac{1}{\mu}\tau_{xy}.$

(3) 均匀压缩试验 (测定E或$\nu$).

$\sigma_x=\sigma_y=\sigma_z=-P$(常数) $\tau_{yz}=\tau_{zx}=\tau_{xy}=0.$

压缩模数 $K=-\frac{P}{\theta}=-\frac{P}{e_x+e_y+e_z}$. K越大越难压缩.

$\lambda,\mu,E,K,\nu$ 中只有2个相互独立.

当$\nu=\frac{1}{2}$时 $K=\infty$, 此时 $\mu=\frac{E}{3}$, 材料不可压缩.

〈4〉验证.

1) 拉伸(或弯曲)测定E 2) 扭转测定$\mu$ 3) 压缩测定K(或$\nu$)

由于只有2个相互独立 可以互相验证.

五、弹性体力学的~~基本~~问题的建立

5.1 弹性体力学基本方程

(1) 力学方程（平衡方程）—纳维方程

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \left(= \rho \frac{\partial^2 u}{\partial t^2}\right) \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \left(= \rho \frac{\partial^2 v}{\partial t^2}\right) \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \left(= \rho \frac{\partial^2 w}{\partial t^2}\right) \end{cases}$$ 5-1

(2) 几何方程（柯西方程）

$$\begin{cases} e_x = \frac{\partial u}{\partial x} \\ e_y = \frac{\partial v}{\partial y} \\ e_z = \frac{\partial w}{\partial z} \\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{cases}$$ 5-2

变形的连续性方程/应变协调方程/圣维南方程

$$\begin{cases} \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \\ \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 e_x}{\partial y \partial z} \\ \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 e_y}{\partial x \partial z} \\ \cdots \end{cases}$$ 5-3



(3) 应力应变关系

a. 用应力表示应变的关系

$$\begin{cases} e_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \\ e_y = \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \\ e_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) \\ \gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz} \\ \gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx} \\ \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \end{cases}$$

$\theta = \frac{1-2\nu}{E}\Theta$ 5-4a

b. 用应变表示应力—拉梅方程

$$\begin{cases} \sigma_x = \lambda\theta + 2\mu e_x \\ \sigma_y = \lambda\theta + 2\mu e_y \\ \sigma_z = \lambda\theta + 2\mu e_z \\ \tau_{yz} = \mu\gamma_{yz} \\ \tau_{zx} = \mu\gamma_{zx} \\ \tau_{xy} = \mu\gamma_{xy} \end{cases}$$

~~5-4b~~ 5-5a

(4) 边界条件和初始条件

力边界条件：
$$\begin{cases} X_N = \sigma_x l + \tau_{xy} m + \tau_{zx} n \\ Y_N = \tau_{xy} l + \sigma_y m + \tau_{yz} n \\ Z_N = \tau_{zx} l + \tau_{yz} m + \sigma_z n \end{cases}$$ 5-6a

DATE:

若为位移边界 $\begin{cases} u=u^* \\ v=v^* \\ w=w^* \end{cases}$ 5-6b

初始条件:

$t=0$时 $\begin{cases} u=f_1(x,y,z) & v=f_2(x,y,z) & w=f_3(x,y,z) \\ \frac{\partial u}{\partial t}=\varphi_1(x,y,z) & \frac{\partial v}{\partial t}=\varphi_2(x,y,z) & \frac{\partial w}{\partial t}=\varphi_3(x,y,z). \end{cases}$ 5-6c

共有 $\underbrace{15\text{个微分方程}}_{\downarrow}$ + 3个边界条件 + 6个初始条件

(5-1), (5-2) (5,4)或(5-5a)

Beltrami-Michell 方程:

$$\begin{cases} \nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} + \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)+2\frac{\partial f_x}{\partial x}=0 \\ \qquad \left(=\frac{2(1+\nu)}{E}\rho\frac{\partial^2\sigma_x}{\partial t^2}-\frac{\rho\nu}{(1-\nu)E}\frac{\partial^2\Theta}{\partial t^2}\right) \\ \cdots \\ \cdots \\ \cdots \\ \nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}+\left(\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)=0 \\ \qquad \left(=\rho\frac{2(1+\nu)}{E}\frac{\partial^2}{\partial t^2}\tau_{yz}\right) \\ \cdots \end{cases}$$



DATE:

求解弹性力学问题的两种基本方法: 位移解法和应力解法.

~~应变法和方力法~~

位移解法以~~应力~~位移分量为基本未知量进行求解 $\begin{cases}$ 求解 $\begin{cases}$ 平衡微分方程(纳维) \\ 以位移表示的平衡微分方程(拉梅方程). $\end{cases}$ $\end{cases}$

应力 ⋯ 应力 ⋯ $\begin{cases}$ 求解 (5-1)(纳维) \\ 和 由 Beltrami-Michell 方程. $\end{cases}$

5.5 双调和函数.

双调和函数: $\begin{cases}$ 在区域内连续 且 对于 $x,y,z$ 的四阶偏微商连续 \\ 其 $\chi$ 在区域内 $\nabla^4\chi=\frac{\partial^4\chi}{\partial x^4}+\frac{\partial^4\chi}{\partial y^4}+\frac{\partial^4\chi}{\partial z^4}+2\frac{\partial^4\chi}{\partial x^2\partial y^2}+2\frac{\partial^4\chi}{\partial y^2\partial z^2}+2\frac{\partial^4\chi}{\partial z^2\partial x^2}=0$. $\end{cases}$

弹性静力学问题中: 位移, 应力, 应变 分量都是双调和函数.

$\nabla^4 u=0$ . . . —,

P93 5.6.

~~$\nabla\cdot F_x$~~ $\nabla U$ 梯度 $\nabla\cdot U$ 散度 $\nabla\times U$ 旋度

~~$\nabla\times F_x$~~ ~~$\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}$~~ $\nabla U=\frac{\partial U}{\partial x}\vec{i}+\frac{\partial U}{\partial y}\vec{j}+\frac{\partial U}{\partial z}\vec{k}$

~~$\nabla\cdot\nabla x$~~ $\nabla\cdot U=\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}$ 描述通量强度

$$\nabla\times U=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U_x & U_y & U_z \end{vmatrix}=\left(\frac{\partial U_z}{\partial y}-\frac{\partial U_y}{\partial z}\right)\vec{i}+\cdots\cdots$$

描述环流量强度.

高斯定理：$\iiint_V \nabla\cdot\vec{F}\,dV=\oiint_S \vec{F}\cdot ds$、

矢量的旋度的散度为0：$\nabla\cdot(\nabla\times\vec{F})=0$.





六 弹性力学的一般原理、

6.1 应变能原理（Clapeyron定理）

定理：弹性体在外力作用下平衡时，变形的弹性能或应变能等于外力对弹性体各点从原有位置经过一位移到达平衡位置时所作的功。假如所加的外力是从0变到指定的值，而在过程进行之中的每一步，物体都处在平衡状态。

6.2 唯一性定理（Kirchhoff定理）

~~受确定体力~~ 定理：假如弹性体受已知的体力作用，在边界面上的外力或是位移是已知的，则弹性体在平衡时，体内各点的应力、应变分量的解是唯一的。

6.3 互换定理（Betti）

定理：第一组力（包括外力及惯性力）在第二组位移上所作的功，等于第二组力（包括外力及惯性力）在第一组位移上所作的功。

6.4 最小势能定理

定理：在适合已知位移边界条件的一切位移中，以适合平衡方程的位移的相关总势能最小。（凡弹性力学中平衡问题的正确解，其相关的总势能为一切适合位移边界条件和连续条件的其他近似解中的最小者）。

## 6.5 虚功原理.

设一弹性体在已知的体力和面力的作用下平衡了。若这弹性体的位移增加微小的一部，则外力便要由于这些微小的位移(虚位移)而做功。这些功等于应变能的增加部分。

## 6.6 最小余能原理.

在适合平衡方程和应力已知部分的边界条件的应力分量的各种函数中，正确的应力分量解，一定能使余能 $V^*$ ~~$(V^*$~~ 为最小值。

$$V^*(\sigma)=U(\sigma)-\iint_{S_\sigma}(u^*\bar{X}_N+v^*\bar{Y}_N+w^*\bar{Z}_N)dS.$$

## 6.7 卡斯提也努定理